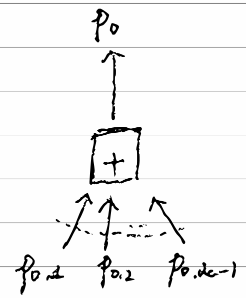

Let $p_{0,i}$ $(i=1,2,\dots,d_c-1)$ be the probability of 0 of ith input message.
p_0 be the probability of 0 of output message.

Note that: $[p_0 \;\; p_1] = [p_{0,1} \;\; p_{1,1}] \circledast [p_{0,2}, \; p_{1,2}] \circledast \cdots \circledast [p_{0,d_c-1} \;.\; p_{1,d_c-1}]$,
where $\circledast$ denotes circular convolution.

Hence:

$$\mathcal{F}[p_0, p_1] = \prod_{i=1}^{d_c-1} \mathcal{F}[p_{0,i}, p_{1,i}]. \qquad (1)$$

where $\mathcal{F}$ is DFT, and multiplication here is entrywise.

Recall DFT:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}\cdot kn}.$$

Hence, the 2-point DFT of $[p_0, p_1]$ is:

$$[\,p_0+p_1 \;,\; p_0 + p_1 e^{-j\frac{2\pi}{2}}\,] = [\,p_0+p_1 \;,\; p_0 - p_1\,]$$

Equation (1) gives:

$$[\underbrace{p_0+p_1}_{=1}, p_0-p_1] = [\underbrace{\prod_{i=1}^{d_c-1}(p_{0,i}+p_{1,i})}_{=1}, \prod_{i=1}^{d_c-1}(p_{0,i}-p_{1,i})]$$

$$\boxed{\therefore\; p_0 - p_1 = \prod_{i=1}^{d_c-1}(p_{0,i}-p_{1,i})}$$

Let $l_0 = \log\frac{p_0}{p_1}$, then: $\frac{p_0}{p_1} = e^{l_0} \Rightarrow p_0 = p_1 e^{l_0} \Rightarrow \begin{cases} p_0 - p_1 = p_1(e^{l_0}-1) \\ p_0+p_1 = p_1(e^{l_0}+1) \end{cases}$

$$\boxed{\therefore\; p_0 - p_1 = \frac{p_0-p_1}{p_0+p_1} = \frac{e^{l_0}-1}{e^{l_0}+1} = \tanh\left(\frac{l_0}{2}\right)} \qquad \left[\tanh(x) = \frac{e^{2x}-1}{e^{2x}+1}\right]$$

$$\therefore \tanh\left(\frac{l_0}{2}\right) = \prod_{i=1}^{d_c-1}\tanh\left(\frac{l_i}{2}\right) \Rightarrow l_0 = 2\cdot\tanh^{-1}\left(\prod_{i=1}^{d_c-1}\tanh\left(\frac{l_i}{2}\right)\right) \qquad \square$$